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Journal of Geometry and Physics 24 (1998) 265–290

JOURNAL OF GEOMETRY AND PHYSICS

Algebraic-geometry approach to integrability of birational plane mappings. Integrable birational quadratic reversible mappings. I

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Received 22 January 1997

Abstract

Using classic results of algebraic geometry for birational plane mappings in plane CP^2 we present a general approach to algebraic integrability of autonomous dynamical systems in C^2 with discrete time and systems of two autonomous functional equations for meromorphic functions in one complex variable defined by birational maps in C^2. General theorems defining the invariant curves, the dynamics of a birational mapping and a general theorem about necessary and sufficient conditions for integrability of birational plane mappings are proved on the basis of a new idea – a decomposition of the orbit set of indeterminacy points of direct maps relative to the action of the inverse mappings. A general method of generating integrable mappings and their rational integrals (invariants) I is proposed. Numerical characteristics N_k of intersections of the orbits $\Phi_n^{-k} O_i$ of fundamental or indeterminacy points $O_i \in \mathbf{O} \cap \mathbf{S}$, of mapping Φ_n, where $\mathbf{O} = \{O_i\}$ is the set of indeterminacy points of Φ_n and $\mathbf{S}$ is a similar set for invariant I, with the corresponding set $\mathbf{O}' \cap \mathbf{S}$, where $\mathbf{O}' = \{O_i'\}$ is the set of indeterminacy points of inverse mapping Φ_n^{-1}, are introduced. Using the method proposed we obtain all nine integrable multiparameter quadratic birational reversible mappings with the zero fixed point and linear projective symmetry $S = CAC^{-1}$, $A = diag(\pm 1)$, with rational invariants generated by invariant straight lines and conics. The relations of numbers N_k with such numerical characteristics of discrete dynamical systems as the Arnold complexity and their integrability are established for the integrable mappings obtained. The Arnold complexities of integrable mappings obtained are determined. The main results are presented in Theorems 2–5, in Tables 1 and 2, and in Appendix A.

Subj. Class.: Dynamical systems
PACS: 02.30.Ks
1991 MSC: 27F07, 34C35
Keywords: Birational mappings; Discrete dynamical systems; Integrability; Rational invariant; Invariant pencil; Algebraic geometry; Reversible mappings

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PII S0393-0440(97)00013-2

1. Introduction

The problem of integrability of birational or Cremona mappings is constantly attracting attention of many researchers already in the course far more than 20 years [1–20]. The interest in this problem stems from the fact that dynamical systems with discrete time defined by such maps arise in very different scientific problems: autonomous reductions of differential–difference soliton equations [2,3], non-algebraic integrable reversible functional equations of static model in the dispersion approach [4–8], quantum integrable systems in lattice statistical mechanics [10–15], discrete versions of integrable systems of classic mechanics [18], integrable lattice nonlinear evolution equations [19,20] and others (see, for example, survey [9]).

As a rule, owing to the existence of some discrete symmetry in systems, the corresponding mappings are reversible dynamical systems which are qualitatively similar to Hamiltonian systems [46–60], although, e.g., the reversible Kolmogorov–Arnold–Moser (KAM) theory possesses some features having no analogues for Hamiltonian systems [60]. The theorems on the existence of the KAM tori in reversible non-Hamiltonian flows [46–51,54–58] and non-symplectic mapping [49–52,58,59] further promote an investigation of the integrability problem of birational mappings.

Recently, some authors have studied k-reversible mappings [61–63], which also may play an important role in various scientific problems. Therefore, a general approach to integrability problem can also be useful for their theory. On the other hand, it seems obvious that the integrability problem and the mapping dynamics are closely related and in this context it is very important to comprehend the dynamics of mappings and to establish the relations between integrability and such numerical characteristic of dynamical systems as the complexity introduced recently by Arnold [43,44].

Attempts to understand integrability of some concrete dynamical systems from the algebraic-geometry point of view [15] or in the framework of the discrete version [16,17] of the Painlevé idea about a moving singularity were undertaken recently.

Note also that integrability of polynomial plane mappings in the subgroup $GA_2 \in BirCP^2$ (or Cr_2) was investigated in papers [9,64,65] and it is very interesting to analyse this problem from a general viewpoint of integrability of birational mappings.

In papers [9,65] an integrability of autonomous dynamical system with discrete time given by bipolynomial mapping in C^2 is defined by means of an existence of a non-trivial commuting map (symmetry of dynamical system). Below, in this paper, we define an integrability or algebraic integrability of autonomous dynamical system with discrete time given by birational mapping in C^2 by means of an existence of a rational first integral or invariant of the dynamical system.

In this paper, using classic results of algebraic geometry for birational mapping in plane CP^2, we find necessary and sufficient conditions for algebraic integrability of autonomous dynamical systems in C^2 with discrete time and systems of two autonomous functional equations for meromorphic functions in one complex variable defined by birational mappings in C^2, we present the method of obtaining their rational first integrals and also obtain the equations of the dynamics of birational mappings. We set the relation of some new numerical characteristics of dynamics of mappings with the integrability and the Arnold complexity. We also present the method of generating integrable plane mappings and on the basis of this method we obtain nine integrable multiparameter quadratic birational reversible mappings with zero fixed point. An important role in our approach belongs to a new concept of the decomposition of the set of indeterminacy points of birational mapping and the set of their orbits. Thus, whereas in papers [4,5] and [6–8] we established very interesting relations between the non-algebraic integrability of some functional equations, defined by birational mappings in the group BirCP^n, and classic results [38,39] in the theory of dynamical systems and in the transcendental number theory [40,41] (see also [42] where were also used the famous results [40,41]), respectively, in this paper we establish a deep relation of the algebraic integrability problem with the algebraic geometry and solve the problem.

In Section 2, we reduce the problem of algebraic integrability of autonomous dynamical systems in C^2 with discrete time and systems of autonomous functional equations for two meromorphic functions in one complex variable to a finding of a rational invariant for corresponding birational mapping in CP^2.

Then in Section 3, we present a necessary brief review of the main definitions and results of the theory of mappings in the group BirCP^2 given in monograph [24]. In Section 4, we formulate a theorem on invariant curves, introduce a new concept of the decomposition of the set of indeterminacy points of a mapping and that of the set of their orbits, prove a theorem on dynamics of mappings and the central theorem of the paper on integrability of birational mappings and on this basis propose a general method of generating integrable of birational plane mappings.

In Section 5, we apply this method to quadratic birational reversible mappings and generate all nine integrable maps with invariant straight lines and conics, the explicit forms of which with invariants in the triangular and usual basis are given in Appendix A. The results of the dynamical studies of these maps are presented in Tables 1 and 2, where are also given the numbers N_k, related with the complexity by Arnold and having the meaning of the sublevels of the complexity.

In the end, in conclusion, Section 6, we briefly discuss a relation of our results with the famous results of Kantor [28–31] and Wiman [34] in the finite subgroups of the Cremona group BirCP^2 and results of M. Noether, E. Bertini, G. Castelnuovo and S. Kantor in the birational classification of linear systems of algebraic curves (see [26,37, Theorem 7.4]), which like [26,27], became known to the author due to discussions with M.Kh. Gizatullin, V.A. Iskovskikh and A.N. Tyurin, when this paper was finished.

In the subsequent paper, part II of this paper, we obtain, within this method, all integrable quadratic reversible mappings with a zero fixed point and with invariants generated by invariant cubics and study their dynamics. In the next paper we will consider the local and global integrability and non-integrability of the known cubic polynomial Moser mapping [64] in the framework of our approach.

2. Formulation of the problem

Let $z = (z_1, z_2, z_3)$ be a point of projective plane CP^2 and mapping $\Phi_n : CP^2 \to CP^2$

$$\Phi_n : z \mapsto z' = z_1' : z_2' : z_3' = \phi_1(z) : \phi_2(z) : \phi_3(z)$$

is a birational one (inverse mapping is also rational), where $\phi_i(z)$ are homogeneous polynomials of degree n in z.

Let us introduce $x \in C^2$, $x_i = z_i/z_3$, $i = (1, 2)$ and consider an autonomous dynamical system in C^2 with discrete time

$$x_i(n+1) = \frac{\phi_i(x_1(n), x_2(n), 1)}{\phi_3(x_1(n), x_2(n), 1)}, \quad i = (1, 2), \tag{1}$$

and a system of autonomous functional equations for meromorphic functions $x_i(w)$, $x \in C^2$, $w \in C$, $i = (1, 2)$

$$x_i(w+1) = \frac{\phi_i(x_1(w), x_2(w), 1)}{\phi_3(x_1(w), x_2(w), 1)}, \quad i = (1, 2). \tag{2}$$

Let us call the systems (1) and (2) algebraically integrable if there exists a mapping $C^2 \to C$ defined by a ratio $I_\mu(x) = g_\mu(x)/h_\mu(x)$ of two polynomials of degree μ, which is invariant with respect to the change $n \to n+1$ or $w \to w+1$:

$$I_\mu(x(n+1)) = I_\mu(x(n)), \qquad I_\mu(x(w+1)) = I_\mu(x(w)).$$

Then the equations

$$I_\mu(x(n)) = c_1, \quad c_1 = \text{const}, \qquad I_\mu(x(w)) = \alpha(w), \quad \alpha(w+1) = \alpha(w), \tag{3}$$

defining the level lines of first integral or invariant of dynamical systems (1) and (2), give one-parameter family or pencil of algebraic curves of degree μ due to a rationality $I_\mu(x)$. Since algebraic curves of genus g are parametrized by rational substitutions at $g = 0$, the elliptic functions at $g = 1$ and the theta-functions of genus g at $g \geq 2$ that, thus, we obtain general solutions of systems (1) and (2) in the form

$$x_i(n) = F_i(n + c_2, c_1), \qquad x_i(w) = F_i(w + \beta(w), \alpha(w)),$$

where $\beta(w)$ is an another arbitrary function in w with a period equal to 1, but the constant c_2 defines a point of reference on the level line (3). In [4–7] we investigated some non-algebraic integrable quadratic birational functional equations of the form (2) with a holomorphic invariant $I_\mu(x)$.

Below we are intended to find the conditions of existence of a rational invariants of birational mappings in CP^2 and to present the method of obtaining them.

3. Some facts from the theory of birational mappings

Cremona mappings are birational self-maps of the n-dimensional projective space kP^n over field k, for $n \geq 2$, their systematic study in the case $n = 2$ and $k = C$ was began by the

Italian geometer M. Cremona in the second half of the 19th century. From the algebraic point of view, a Cremona map is a k-automorphism of the rational function field $k(z_1, z_2, \ldots, z_n)$ in n variables, for some $n \geq 2$. [2]

The main tool for studying birational mappings is the technique of linear systems with assigned base conditions in dimension 2; the most complete modern treatment of them is presented in monograph [22]. Below we shall follow monograph [24] (see also [21–23,25–27]).

Definition-Theorem 1. Let $z = (z_1, z_2, z_3)$ be a point of projective plane CP^2. A mapping $\Phi_n : \mathrm{CP}^2 \to \mathrm{CP}^2$

$$\Phi_n : z \mapsto z' = z'_1 : z'_2 : z'_3 = \phi_1(z) : \phi_2(z) : \phi_3(z), \tag{4}$$

where ϕ_i are homogeneous polynomials in z, $i = (1, 2, 3)$, of degree n, is called a birational mapping if it assigns one-to-one correspondence between z and z', while the inverse mapping is given by

$$\Phi_n^{-1} : z' \mapsto z = z_1 : z_2 : z_3 = \phi'_1(z') : \phi'_2(z') : \phi'_3(z'), \tag{5}$$

and is also rational, ϕ'_i being also homogeneous polynomials in z', moreover, ϕ_i and ϕ'_i have no common factors. Associated with Φ_n is the linear system $\phi = c_1\phi_1 + c_2\phi_2 + c_3\phi_3$ (for $c_i \in \mathrm{C}$). One-to-one correspondence for direct Φ_n and inverse Φ_n^{-1} mappings is not fulfilled only at indeterminacy or fundamental points $O_\alpha \in \mathbf{O}$, $O'_\beta \in \mathbf{O}'$, $\alpha, \beta = (1, 2, \ldots, \sigma)$, i.e., common zeros of multiplicities i_α, i'_β for functions $\phi_k(z), \phi'_k(z)$, $k = (1, 2, 3)$, and the associated linear systems ϕ, ϕ' (below we suppose without loss of generality that the coordinate z_3 of O_α and O'_β are not equal to zero)

$$\left.\frac{\partial^l \phi_k(z)}{\partial z_1^{l-m}\partial z_2^m}\right|_{O_\alpha} = 0 \quad \text{for } l = 0, 1, 2, \ldots, i_\alpha - 1, \ \ 0 \leq m \leq l,$$

$$\left.\frac{\partial^l \phi'_k(z)}{\partial z_1^{l-m}\partial z_2^m}\right|_{O'_\beta} = 0 \quad \text{for } l = 0, 1, 2, \ldots, i'_\beta - 1, \ \ 0 \leq m \leq l,$$

and on principal or exceptional curves J_α, J'_β, $\alpha, \beta = (1, 2, \ldots, \sigma)$,

$$J_\alpha \stackrel{\text{def}}{=} \{z: j_\alpha(z) = 0\}, \qquad J'_\beta \stackrel{\text{def}}{=} \{z: j'_\beta(z) = 0\}, \quad \alpha, \beta = (1, 2, \ldots, \sigma),$$

where j_α, j'_β are homogeneous polynomials in z of degrees i_α, i'_β, respectively, moreover, points O_α, O'_β blow up into curves J'_α, J_β of degrees i_α, i'_β and curves J_α, J'_β blow down into points O'_α, O_β, respectively (see the concept of σ-process of blowing up of singularities in the theory of ordinary differential equations [66] and the Kodaira theorem in the algebraic geometry [23]). The multiplicity of a fundamental point is the multiplicity at this point of a

[2] From a modern Foreword [25] to monograph [24] by V.A. Iskovskikh and M. Reid in connection with a new edition of the book planned in the future.

general curve ϕ. In special cases tangency conditions of any two members of the associated linear system are expressed as multiplicities of infinitely near points [25], or adjoint points in the therminology of the Hudson book. The Jacobian J of the mapping Φ_n equals

$$J = \left\| \frac{\partial \phi_k}{\partial z_i} \right\| = \prod_{\alpha=1}^{\sigma} j_\alpha .$$

The determination of the Jacobian is a very simple way to find the principal curves. The principal curves J_α (J'_β) intersect each other only in fundamental points O_α (O'_β).

If we substitute z from (5) into (4), we obtain the identity

$$z'_1 : z'_2 : z'_3 \equiv \phi_1(\phi'(z')) : \phi_2(\phi'(z')) : \phi_3(\phi'(z')),$$

and there is a factor of proportionality

$$G'(z') \equiv \frac{\phi_i(\phi'(z'))}{z'_i} \quad \text{for all } i \in (1, 2, 3),$$

where $G'(z')$ is a homogeneous polynomial in z' of degree $n^2 - 1$.

Linear combinations of the functions ϕ_i, ϕ'_i

$$\phi \equiv c_1\phi_1 + c_2\phi_2 + c_3\phi_3, \qquad \phi' \equiv c'_1\phi'_1 + c'_2\phi'_2 + c'_3\phi'_3$$

define the first and second rational nets which are the images of nets of lines. The curves $\phi = 0$, $\phi' = 0$ are rational (of genus $g = 0$).

Remark 1. Note that for a polynomial mapping the functions $\phi_3(z)$, $\phi'_3(z)$ are identically equal to z_3^n (see, for example, the known Moser cubic mapping [64]).

Remark 2. The transition from CP^2 to C^2 is given by the change $z \to x$, $x \in C^2$, $x_i = z_i/z_3$, $i \in (1, 2)$, and $x'_i = \phi_i(xz_3, z_3)/\phi_3(xz_3, z_3)$.

Remark 3. The set of numbers $n; i_1, i_2, \ldots, i_\sigma$, $i_1 \geq i_2 \geq \cdots \geq i_\sigma$, where i_α are the multiplicities of all the F-points of Φ_n, including infinitely near ones , is called the characteristic of mapping Φ_n. The general mapping with a given characteristic depends on $2\sigma + 8$ parameters. If the characteristic of the inverse mapping is the same, then this characteristic is called self-conjugate; otherwise, it is called conjugate. For n general, there are always at least two self-conjugate characteristics. There are the following inequalities for $n \geq 2$ (the latest one is the Noether inequality):

$$\sigma \leq 2n - 1, \quad i_1 + i_2 \leq n, \quad i_1 + i_2 + i_3 \geq n + 1.$$

All characteristics up to $n = 17$ are in [24]. At $n = 3$ and $n = 4$ they are

$$n = 3: \quad 3; 2, 1, 1, 1, 1, \qquad n = 4: \quad 4; 3, 1, 1, 1, 1, 1, 1, \quad 4; 2, 2, 2, 1, 1, 1.$$

Remark 4. Let $i'_{\beta\alpha}$ be the multiplicity of curve J'_α at point O'_β and $i_{\alpha\beta}$ be that of curve J_β at O_α. Then we have the equality $i_{\alpha\beta} = i'_{\beta\alpha}$ and the following relations between numbers $i_\alpha, i'_\beta, i_{\alpha\beta}$, expressing certain geometrical facts (summing in the left column over α and in the right one over β from 1 to σ):

$$\sum i_\alpha = 3(n-1), \qquad \sum i'_\beta = 3(n-1), \tag{6}$$

$$\sum i_\alpha^2 = n^2 - 1, \qquad \sum {i'_\beta}^2 = n^2 - 1, \tag{7}$$

$$\sum i_{\alpha\beta} = 3i'_\beta - 1, \qquad \sum i_{\alpha\beta} = 3i_\alpha - 1, \tag{8}$$

$$\sum i_\alpha i_{\alpha\beta} = i'_\beta n, \qquad \sum i'_\beta i_{\alpha\beta} = i_\alpha n, \tag{9}$$

$$\sum i_{\alpha\beta}^2 = i'^2_\beta + 1, \qquad \sum i_{\alpha\beta}^2 = i_\alpha^2 + 1, \tag{10}$$

$$\sum i_{\alpha\beta} i_{\alpha\gamma} = i'_\beta i'_\gamma \quad (\beta \neq \gamma), \qquad \sum i_{\alpha\beta} i_{\gamma\beta} = i_\alpha i_\gamma \quad (\alpha \neq \gamma). \tag{11}$$

Remark 5. Consider properties of a general curve $f_\mu(z') = 0$ of degree μ under the mapping (4). By map (4), the curve $f_\mu(z')$ is mapped into curve $f_\mu(\phi(z)) = f'_{\mu'}(z)$ of degree $\mu' = \mu n$, moreover, every point O_α which is i_α-fold on $\phi(z)$ is μi_α-fold on $f'_{\mu'}$. If $f_\mu(z')$ has multiplicities γ'_β at points O'_β, then ($\deg(j_\beta) \equiv i'_\beta$)

$$f_\mu(z') = f'_{\mu'}(z) \prod_{\beta=1}^{\sigma} j_\beta^{\gamma'_\beta}, \quad \mu' = \mu n - \sum_{\beta=1}^{\sigma} \gamma'_\beta i'_\beta, \tag{12}$$

moreover, $f'_{\mu'}$ has multiplicities γ_α at O_α (see the meaning of $i_{\alpha\beta}$ in Remark 4):

$$\gamma_\alpha = \mu i_\alpha - \sum_{\beta=1}^{\sigma} i_{\alpha\beta} \gamma'_\beta. \tag{13}$$

Remark 6. Let $R = \sum \frac{1}{2}\rho_\nu(\rho_\nu - 1)$ be the reduction of genus of a linear family of curves $\{f_\mu\}$ due to all its σ_1 ρ_ν-fold points $S_\nu \in \mathbf{S}$ other than its $\sigma_2 \leq \sigma$ γ_α-fold points from $\mathbf{O}$; these are mapped onto the multiple points S'_δ of linear family $\{f'_{\mu'}\}$ other than $\mathbf{O}'$, reducing the genus of $f'_{\mu'}$ by R also; let q be apparent freedom of curves $\{f_\mu\}$, that is, the one calculated under the assumption that all the base points impose independent conditions on $\{f_\mu\}$:

$$q = \frac{1}{2}\mu(\mu+3) - \sum_{\nu=1}^{\sigma_1} \frac{1}{2}\rho_\nu(\rho_\nu + 1) - \sum_{\alpha=1}^{\sigma_2} \frac{1}{2}\gamma_\alpha(\gamma_\alpha + 1), \tag{14}$$

then the relation

$$\begin{aligned} p &= \frac{1}{2}(\mu - 1)(\mu - 2) - \sum \frac{1}{2}\gamma_\alpha(\gamma_\alpha - 1) - R \\ &= \frac{1}{2}(\mu' - 1)(\mu' - 2) - \sum \frac{1}{2}\gamma'_\beta(\gamma'_\beta - 1) - R \end{aligned} \tag{15}$$

expresses the invariance of genus p of curves f_μ.

Definition-Theorem 2. The set of fixed points $\{D_l\}$ of mapping Φ_n (4) is defined as the intersection of two $(n+1)$-ics (the conic ≡ the 2-ic, the cubic ≡ the 3-ic, and so on)

$$f_1 = z_1\phi_3 - z_3\phi_1, \quad f_2 = z_2\phi_3 - z_3\phi_2.$$

The intersections of those which are not invariant are the F-points O_α and the n intersections of z_3 and ϕ_3. In general, therefore, the number of isolated points D_l is $n+2$. If the two $(n+1)$-ics have a common part, consisting of fixed points only, and there is the fixed (invariant) curve, say Δ_μ of degree $\mu \le \frac{2}{3}(n+1)$, then the number of isolated fixed points is reduced. There are simple fixed points at a simple intersection f_1, f_2, simple fixed points of s-point contact and i-fold points.

Theorem 1 (M. Noether). *Every Cremona plane mapping can be resolved into quadratic mappings.*

Remark 7. The procedure of resolution of mapping $\Phi_n \to \Phi_{n'}\circ\Phi_2$ into two simpler components $\Phi_{n'}$, Φ_2 is not unique since any Φ_2 can be replaced by two others having two F-points in common; hence any set of Φ_2 is equivalent to an infinite number of other sets. However, the normal resolution is unique and it is defined by the choice of three F-points of Φ_2 being common with three F-points (*top trio* of maximal multiplicities i_1, i_2, i_3) of mapping Φ_n. Then n' is equal to $2n - i_1 - i_2 - i_3 < n$ due to the Noether inequality (see Remark 3). After a series of such resolutions n' will be equal to 2 and resolution is complete. It is obvious that Φ_2 cannot be resolved in Φ_1. Let us note that, if the top trio is on direct line, then the Noether method fails, but J.W. Alexander corrected the Noether theorem [26] in this case.

Remark 8. Any generic quadratic Cremona mapping is generated by a composition

$$\Phi_2 \equiv B^{-1}\circ I_s \circ B_1, \tag{16}$$

where

$$B : z \mapsto j' = Bz, \qquad B_1 : z \mapsto j = B_1 z \tag{17}$$

are general linear mappings from the PGL(2, C) group and I_s is the involutive standard Cremona mapping with three simple F-points in (1,0,0), (0,1,0) and (0,0,1) and three principal lines $J_\alpha = \{(z_1 = 0), (z_2 = 0), (z_3 = 0)\}$:

$$I_s : z \mapsto z' = z_1' : z_2' : z_3' = z_2 z_3 : z_1 z_3 : z_1 z_2. \tag{18}$$

In the triangular frame of reference (17) mapping (16) takes a very simple form

$$\begin{aligned}\Phi_2 : j(z) \mapsto j'(z') &= j_1'(z') : j_2'(z') : j_3'(z') \\ &= j_2(z) j_3(z) : j_1(z) j_3(z) : j_1(z) j_2(z).\end{aligned} \tag{19}$$

The mapping Φ_2 is specialized if two or three F-points are adjacent or infinitely near [25] and has, respectively, the following forms:

$$\Phi_{2a} \equiv B^{-1} \circ I_a \circ B_1, \qquad I_a : z \mapsto z' = z_1' : z_2' : z_3' = z_2^2 : z_1 z_2 : z_1 z_3, \tag{20}$$

$$\Phi_{2b} \equiv B^{-1} \circ I_b \circ B_1, \qquad I_b : z \mapsto z' = z_1' : z_2' : z_3' = z_1^2 : z_1 z_2 : (z_2^2 - z_1 z_3), \tag{21}$$

moreover, involutions I_a, I_b from (20) and (21) can be resolved as a composition of two or four, but not fewer, general mappings (16), respectively. Any two members of the net (20) touch one another and have a fixed common tangent $j_1 \equiv z_1 = 0$, but ones of the net (21) have fixed common tangent $j \equiv z_1$ and osculate a fixed conic $z_2^2 - z_1 z_3$. These tangency conditions are simulated by two or three infinitely near points, so as Eqs. (6)–(10) remain correct.

4. Main theorems of algebraic integrability of birational plane maps

From Remarks 5 and 6 the following theorem follows.

Theorem 2. *For a plane curve $f_\mu(z) = 0$ of degree μ and genus p, defined by formula* (15), *the following two conditions are equivalent: (A) $f_\mu(z) = 0$ is invariant under the mappings Φ_n* (4) *of characteristic $n; i_1, \ldots, i_\sigma$ (see Remarks* 3 *and* 4) *and Φ_n^{-1}* (5); *(B) $f_\mu(z)$ is a solution of the following functional equations (see* (12)–(15) *in Remarks* 5 *and* 6)

$$f_\mu(\phi(z)) = |\epsilon| sgn(\epsilon) f_\mu(z) \prod_{\beta=1}^{\sigma} j_\beta^{\gamma_\beta'}, \qquad \sum_{\beta=1}^{\sigma} \gamma_\beta' i_\beta' = \mu(n-1), \tag{22}$$

$$f_\mu(\phi'(z)) = |\epsilon|^{-1} sgn(\epsilon) f_\mu(z) \prod_{\alpha=1}^{\sigma} j_\alpha'^{\gamma_\alpha}, \qquad \sum_{\alpha=1}^{\sigma} \gamma_\alpha i_\alpha = \mu(n-1), \tag{23}$$

where O_α, O_β' are γ_α-fold and γ_β'-fold points of the curve $f_\mu(z) = 0$, and multiplicities γ_α and γ_β' satisfy the equation

$$\gamma_\alpha = \mu i_\alpha - \sum_{\beta=1}^{\sigma} i_{\alpha\beta} \gamma_\beta' \tag{24}$$

(about i_α, $i_{\alpha\beta}$ see Definition-Theorem 1 *and Remark* 4), *while $sgn(\epsilon) = \pm 1$ and number ϵ is a constant.*

The number q of free parameters of the curve $f_\mu(z) = 0$ before the substitution into the functional equations (22)–(23) *is given by formula* (14) *(see Remark* 6). *If q equals* 1 *then we shall obtain an invariant pencil, at $q = 2$ or* 3 *we shall find an invariant net or web.*

Let us give a definition of an integrability or algebraic integrability of the mapping Φ_n (4).

Definition 1. The mapping Φ_n (4) is integrable or algebraically integrable if there exists an invariant rational function of z

$$I_\mu(z) = g_\mu(z)/h_\mu(z), \qquad I_\mu(\phi(z)) = I_\mu(z),$$

moreover, equation $I_\mu(z) = c = \text{const}$ defines the level lines of the first integral or invariant $I_\mu(z)$. Note that this definition is equivalent to the existence of invariant one-parameter family or invariant pencil of curves $f_\mu(z) = ag_\mu(z) + bh_\mu(z)$ satisfying Eqs. (22) and (23) moreover, the homogeneous polynomials of degree μ $g_\mu(z), h_\mu(z)$ are any two linear independent solutions of Eqs. (22) and (23) with the same set $\gamma_\alpha, \gamma'_\beta$.

Definition 2. The orbit $\mathcal{O}_z$ of a point z with respect to the mapping Φ_n^{-1} (5) is the set of points $\mathcal{O}_z^k = \Phi_n^{-k}(z) = (\Phi_n^{-1})^k(z)$, $k \in \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of non-negative integers. The orbit $\mathcal{O}'_z$ of a point z with respect to Φ_n (4) is defined analogously, $\mathcal{O}'^k_z = \Phi_n^k(z) = (\Phi_n)^k(z)$ and $\Phi_n^k(z) \overset{\text{def}}{=} \Phi_n(\Phi_n(\cdots(z)\cdots))$, $\Phi_n^{-k}(z) \overset{\text{def}}{=} \Phi_n^{-1}(\Phi_n^{-1}(\cdots(z)\cdots))$(see, for example, [66,68,74]).

Definition 3. If the number k of points of the orbit $\mathcal{O}_z$ of a point z with respect to the mapping Φ_n^{-1} (5) is finite where k is defined by the condition

$$\mathcal{O}_z^k = \Phi_n^{-k}(z) = z, \quad k \in \mathbb{Z}^+,$$

then the periodic points $(\Phi_n^{-m}(z))$, $m = (0, 1, \ldots, k-1)$, form a cycle of index k or period k of the mapping $\Phi_n^{-1}(z)$, but z is a fixed point of the mapping $\Phi_n^{-k}(z)$. A cycle of index k of the mapping $\Phi_n(z)$ (4) is defined similarly with the changes: $\mathcal{O}_z \mapsto \mathcal{O}'_z$ and $\Phi_n^{-m}(z) \mapsto \Phi_n^m(z)$ (see [68]). The number π_m of cyclic sets of order $m = p_1^{m_1} p_2^{m_2} \cdots p_\nu^{m_\nu}$ is given by the Kantor formula [26]:

$$\pi_m = \frac{1}{m}\left(n^m - \sum n^{m/p_i} + \sum n^{m/p_i p_j} + \cdots + (-1)^\nu n^{m/p_1 p_2 \cdots p_\nu}\right) \tag{25}$$

Definition-Theorem 3. Let Φ_n (4) be a mapping of characteristic n; $i_1, i_2, \ldots, i_\sigma$ and Φ_n^{-1} (5) be the inverse mapping (see Definition-Theorem 1, Remark 3). Define the decomposition of the sets $\mathbf{O}, \mathbf{O}'$ of fundamental points O_α, O'_β of these mappings as follows:

$$\mathbf{O} \equiv \mathbf{O}^{(\text{cyc})} \cup \mathbf{O}^{(\text{int})} \cup \mathbf{O}^{(\text{inf})}, \quad \mathbf{O}' \equiv \mathbf{O}'^{(\text{cyc})} \cup \mathbf{O}'^{(\text{int})} \cup \mathbf{O}'^{(\text{inf})}. \tag{26}$$

Here $\mathbf{O}^{(\text{inf})}$ is the subset of fundamental points O_α with infinite orbits, $\mathbf{O}^{(\text{cyc})}$ the subset of fundamental points O_α having cyclic orbits $\mathcal{O}_z^m$, $z \in \mathbf{O}$, of index m_α or cyclic orbits with tail, i.e. $\mathcal{O}_x^{l+m} = \mathcal{O}_z^m = z, x, z \in \mathbf{O}, x \neq z$, and $\mathbf{O}^{(\text{int})}$ is the subset of fundamental points O_α, whose orbits $\mathcal{O}_z^k$, $z = O_\alpha \in \mathbf{O}$, intersect the set $\mathbf{O}'$ and finish for some $k = k_{\alpha\beta}$ by points O'_β ($O_\alpha \equiv O'_\beta$ at $k_{\alpha\beta} = 0$).

Introduce numbers N_k as numbers of intersections of the set $\mathcal{O}_{\mathbf{O}}^k$ of orbits $\mathcal{O}_z^k$, $z \in \mathbf{O}^{(\mathbf{int})}$, with the set $\mathbf{O}'^{(\mathbf{int})}$ (see Definition 2):

$$N_k = \#(\mathcal{O}^k_{\mathbf{O}^{(\mathbf{int})}} \cap \mathbf{O}'^{(\mathbf{int})}), \tag{27}$$

where #**A** denotes the number of points of the set **A**. The decomposition of the set $\mathbf{O}'$ with respect to the action of the mapping Φ_n is entirely analogous. It is obvious that

$$\sum_{k=k_{\min}}^{k=k_{\max}} N_k = \#\mathbf{O}^{(\mathrm{int})} \equiv \#\mathbf{O}'^{(\mathrm{int})}.$$

Definition-Remark 1. Arnold [43,44] introduced and investigated such characteristic of a dynamical system as the topological *complexity* of the intersection of a submanifold, moved by a dynamical system, with a given submanifold of the phase space. In the simplest case for plane mappings Φ the complexity $C_A^{\Phi}(k)$ can be defined [45] as the number of intersection points of a fixed curve Γ_1 with the image of another curve Γ_2 under the kth iteration of Φ:

$$C^{\Phi}_{A:\Gamma_1\Gamma_2}(k) = \#(\Gamma_1 \cap \Phi^k(\Gamma_2)).$$

If the mapping Φ is a birational one in $\mathbf{BirCP}^2$ and the curves Γ_1, Γ_2 are algebraic curves in $\mathbf{CP}^2$, then it is easy to see that the growth of $C^{\Phi}_{A:\Gamma_1\Gamma_2}(k)$ will in general be as follows:

$$C^{\Phi}_{A:\Gamma_1\Gamma_2}(k) = \deg(\Gamma_1)\deg(\Gamma_2)d_{\Phi}(k) \le \deg(\Gamma_1)\deg(\Gamma_2)(\deg\Phi)^k,$$

where $d_{\Phi}(k) = \deg(\Phi^k)$ is the degree of the mapping Φ^k, which agrees well with general Arnold's results for smooth mappings and diffeomorphisms [43,44].

Theorem 3. *Let $d(k)$ be the degree of the mapping Φ_n^k, the kth iteration of the mapping Φ_n* (4), *the number of points of the set* $\mathrm{O}^{(\mathrm{int})}$ *be no less than zero,* $\#\mathrm{O}^{(\mathrm{int})} \ge 0$, *and $\gamma_\alpha(k)$, $\gamma'_\beta(k)$ be the multiplicities of the curves $\phi_i^{(k)}(z) = 0$, $i = (1,2,3)$, at fundamental points of the direct mapping* (4) *O_α and the inverse one* (5) *O'_β. Then the dynamics of the mapping Φ_n* (4), *Φ_n^k, $k \in \mathbb{Z}^+$, of characteristic $n; i_1, i_2, \dots, i_\sigma$ (see Definition-Theorem* 1, *Remarks* 3 *and* 4) *is completely determined by the following formulae:*

$$\Phi_n^k : z \mapsto z', \quad z'_1 : z'_2 : z'_3 = \phi_1^{(k)}(z) : \phi_2^{(k)}(z) : \phi_3^{(k)}(z), \tag{28}$$

$$\phi_i^{(k)}(z) \equiv \phi_i^{(k)}(O_\alpha^{\gamma_\alpha(k)}, ([\Phi_n^{(-l)}(O_\alpha)]^{\gamma_\alpha(k-l)}, l = 1, \dots, m_{\alpha\beta}), \dots), \tag{29}$$

$$d(k) = nd(k-1) - \sum i'_\beta \gamma'_\beta(k-1), \tag{30}$$

$$\gamma_\alpha(k) = d(k-1)i_\alpha - \sum i_{\alpha\beta}\gamma'_\beta(k-1), \tag{31}$$

moreover,

$$\begin{aligned} d(0) &= 1, \quad d(1) = n, \quad \gamma_\alpha(1) = i_\alpha, \quad \gamma_\alpha(k) = 0 \ \textit{for}\ k \le 0, \\ \gamma'_\beta(k) &= \gamma_\alpha(k - m_{\alpha\beta}) \quad \textit{for all}\ \alpha, \beta \end{aligned} \tag{32}$$

that

$$\Phi_n^{-m_{\alpha\beta}}(O_\alpha) \equiv O'_\beta, \quad O_\alpha \in \mathrm{O}^{(\mathrm{int})}, \qquad O'_\beta \in \mathrm{O}'^{(\mathrm{int})}, \tag{33}$$

and, according to (32) *and* (33),

$$\gamma'_\beta(k) = 0 \quad \textit{for} \quad k \le m_{\alpha\beta}, \qquad \gamma'_\beta(m_{\alpha\beta} + 1) = i_\alpha. \tag{34}$$

Proof. Let us prove the theorem by induction method. Let us consider the kth iteration of the mapping Φ_n (4) as a transformation of the curves $\phi_i^{(k-1)}(z) = 0$ by the action of the mapping

Φ_n (4). Let the numbers $\gamma_\alpha(k-1), \gamma_\alpha(k-1-l), l = 1, \dots, m_{\alpha\beta}$, be the multiplicities of the curves $\phi_i^{(k-1)}(z) = 0$ at the points $(O_\alpha, [\Phi_n^{(-l)}(O_\alpha)] \in \mathcal{O}_{O_\alpha}, l = 1, \dots, m_{\alpha\beta})$ and let $\Phi_n^{(-m_{\alpha\beta})}(O_\alpha) = O'_\beta$, where we indicate explicitly only the points $O_\alpha \in \mathbf{O}^{(\mathrm{int})}$, $O'_\beta \in \mathbf{O}'^{(\mathrm{int})}$. The existence of other points $O_\alpha \in \mathbf{O}^{(\mathrm{cyc})}$, $O_\alpha \in \mathbf{O}^{(\mathrm{inf})}$ (see Definition-Theorem 3) is not essential for the growth of $d(k)$, although their multiplicities are also defined by (30) and (31) only with other conditions $\Phi_n^{(-r_{\alpha\beta})}(O_\alpha) \equiv O_\beta$ for $O_\alpha, O_\beta \in \mathbf{O}^{(\mathrm{cyc})}, l = 1, \dots, r_{\alpha\beta}$, and for $O_\alpha \in \mathbf{O}^{(\mathrm{inf})}, l = 1, \dots, k-1$, in (29) and (33).

Then, according to Remark 5 and Eqs. (12) and (13), we have

$$\phi_i^{(k-1)}(z') = \phi_i^{(k)}(z) \prod_{\beta=1}^{\sigma} j_\beta^{\gamma'_\beta(k-1)}(z)$$

and obtain formulae (30) and (31).

Now prove the equalities (29)–(31) for $k = 1$. Indeed, $d(1) = n, \gamma_\alpha(1) = i_\alpha$, therefore $\gamma'_\beta(0) = 0$ and the proof is completed. □

Now we can state a general proposition about the necessary and sufficient conditions of algebraic integrability of Φ_n (4) (see Definition 1).

Theorem 4. *Let Φ_n (4) be a mapping of characteristic $n; i_1, i_2, \dots, i_\sigma$, Φ_n^{-1} (5) be the inverse mapping of characteristic $n; i'_1, i'_2, \dots, i'_\sigma$ and $i_{\alpha\beta}$ be the multiplicities of curve J_β at O_α (see Definition-Theorem 1 and Remark 3). Accomplish the decomposition of the sets $\mathbf{O}, \mathbf{O}'$ of fundamental points O_α, O'_β with respect to the action of mappings Φ_n^{-1} and Φ_n (see Definition-Theorem 3).*

Then, if the mapping Φ_n (4) is algebraically integrable and $I_\mu(z)$ is its invariant (see Definition 1), the set $\mathbf{S} \equiv (g_\mu(z) = 0) \cap (h_\mu(z) = 0)$ of μ^2 (due to the Bezu theorem) indeterminacy points of multiplicities $\{\gamma_\alpha, \gamma'_\beta, \rho_\nu\} \in$ the set Γ of the invariant $I_\mu(z)$ admits the following decomposition:

$$\mathbf{S} \equiv \mathbf{S}^{(\mathrm{cyc})} \cup \mathbf{S}^{(\mathrm{int})} \cup \mathbf{S}'^{(\mathrm{cyc})} \cup \mathbf{S}'^{(\mathrm{int})} \cup \bar{\mathbf{S}}, \tag{35}$$

$$\bar{\mathbf{S}} \equiv \bar{\mathbf{S}}^{(\mathrm{cyc})} \cup \bar{\mathbf{S}}^{(\mathrm{int})} \cup \bar{\mathbf{S}}'^{(\mathrm{cyc})} \cup \bar{\mathbf{S}}''^{(\mathrm{cyc})}, \tag{36}$$

where

$$\mathbf{S}^{(\mathrm{cyc})} \subseteq \mathbf{O}^{(\mathrm{cyc})}, \quad \mathbf{S}^{(\mathrm{int})} \subseteq \mathbf{O}^{(\mathrm{int})}, \quad \mathbf{S}'^{(\mathrm{cyc})} \subseteq \mathbf{O}'^{(\mathrm{cyc})}, \quad \mathbf{S}'^{(\mathrm{int})} \subseteq \mathbf{O}'^{(\mathrm{int})}, \tag{37}$$

$$\begin{aligned} &\bar{\mathbf{S}}^{(\mathrm{cyc})} \equiv \mathcal{O}_{\mathbf{S}^{(\mathrm{cyc})}} \backslash \mathbf{S}^{(\mathrm{cyc})}, \quad \bar{\mathbf{S}}^{(\mathrm{int})} \equiv \mathcal{O}_{\mathbf{S}^{(\mathrm{int})}} \backslash \mathbf{S}^{(\mathrm{int})} \equiv \mathcal{O}'_{\mathbf{S}'^{(\mathrm{int})}} \backslash \mathbf{S}'^{(\mathrm{int})}, \\ &\bar{\mathbf{S}}'^{(\mathrm{cyc})} \equiv \mathcal{O}'_{\mathbf{S}'^{(\mathrm{cyc})}} \backslash \mathbf{S}'^{(\mathrm{cyc})}, \quad \bar{\mathbf{S}}''^{(\mathrm{cyc})} : \mathcal{O}_{\bar{\mathbf{S}}''^{(\mathrm{cyc})}} \equiv \bar{\mathbf{S}}''^{(\mathrm{cyc})}, \end{aligned} \tag{38}$$

where the expression $A \backslash B$ means a set A without a set B, moreover, the set $\mathbf{S}^{(\mathrm{int})}$ corresponds to the set $\mathbf{S}'^{(\mathrm{int})}$ as the set $\mathbf{O}^{(\mathrm{int})}$ corresponds to the set $\mathbf{O}'^{(\mathrm{int})}$ and $\#\mathbf{S}^{(\mathrm{int})} = \#\mathbf{S}'^{(\mathrm{int})}$, but

$$\mu^2 = \#\mathbf{S}^{(\mathrm{cyc})} + \#\mathbf{S}'^{(\mathrm{cyc})} + 2\#\mathbf{S}^{(\mathrm{int})} + \#\bar{\mathbf{S}}. \tag{39}$$

The sets of multiplicities $\gamma_\alpha, \gamma'_\beta$ correspond to the indeterminacy points from the subsets $\mathbf{S}^{(\mathrm{cyc})} \cup \mathbf{S}^{(\mathrm{int})}, \mathbf{S}'^{(\mathrm{cyc})} \cup \mathbf{S}'^{(\mathrm{int})}$, *but the one ρ_ν corresponds to the indeterminacy points from the subset* $\bar{\mathbf{S}}$.

For the mapping Φ_n (4) *being integrable and having first integral or invariant $I_\mu(z) = g_\mu(z)/h_\mu(z)$, which is equivalent to the existence of a one-parameter family or pencil of invariant curves $f_\mu(z) = a g_\mu(z) + b h_\mu(z)$ of degree μ, the following conditions are necessary and sufficient:*

(1) $\#(\mathbf{O}^{(int)}) \neq 0,$ (40)

(2) *there exists non-trivial set of integers: degree of invariant μ and a set of the multiplicities $\gamma_\alpha, \gamma'_\beta$ satisfying the following equations:*

$$\sum_{\beta=1}^{\sigma} \gamma'_\beta i'_\beta = \mu(n-1), \quad \sum_{\alpha=1}^{\sigma} \gamma_\alpha i_\alpha = \mu(n-1), \quad \gamma_\alpha = \mu i_\alpha - \sum_{\beta=1}^{\sigma} i_{\alpha\beta}\gamma'_\beta, \tag{41}$$

and

$$\sum_{\beta=1}^{\sigma} \gamma'_\beta = \sum_{\alpha=1}^{\sigma} \gamma_\alpha, \qquad \sum_{\beta=1}^{\sigma} {\gamma'_\beta}^2 = \sum_{\alpha=1}^{\sigma} {\gamma_\alpha}^2; \tag{42}$$

(3) *the dimension r of linear system of invariant curves $f_\mu(z)$ with the set S of the basis points determined above by* (35)–(39), (41), (42) *and multiplicities $\gamma_\alpha, \gamma'_\beta$ determined by* (41) *and* (42) *is not less than one, $r \geq 1$, and defined as a number of linearly independent solutions of the functional equations* (22) *and* (23), *reduced by one, and moreover, if their number is equal to $r+1$ then the number of invariants $I_\mu(z)$ equals r and $r-1$ of them depend algebraically on the remaining invariant.*

Note that conditions (1) *and* (2) *are necessary but the one* (3) *is sufficient, moreover, condition* (2) *completely defines the set* **S** *not defined completely by conditions* (35)–(39). *If conditions* (1)–(3) *are fulfilled, then the mapping Φ_n* (4) *is integrable and the invariant $I_\mu(z)$ is of the form $I_\mu(z) = g_\mu(z)/h_\mu(z)$ for functions g_μ, h_μ being a linear independent solutions of the functional equations* (22) *and* (23) *with the same signature $sgn(\epsilon)$ and of the form $I_\mu(z) = [g_\mu(z)/h_\mu(z)]^2$ for the case of different $sgn(\epsilon)$* (*see Theorem* 2).

The genus p of the pencil of invariant curves f_μ and the number q of free parameters of $f_\mu(z)$ before solving the functional equations (22) *and* (23) *are determined by the formulae*

$$p = \frac{1}{2}(\mu-1)(\mu-2) - \sum \frac{1}{2}\gamma_\alpha(\gamma_\alpha - 1) - \sum \frac{1}{2}\gamma'_\beta(\gamma'_\beta - 1) - \sum \frac{1}{2}\rho_\nu(\rho_\nu - 1), \tag{43}$$

$$q = \frac{1}{2}\mu(\mu+3) - \sum \frac{1}{2}\gamma_\alpha(\gamma_\alpha + 1) - \sum \frac{1}{2}\gamma'_\beta(\gamma'_\beta + 1) - \sum \frac{1}{2}\rho_\nu(\rho_\nu + 1). \tag{44}$$

Proof. We will look for the first integral or invariant of degree μ $I_\mu(z) = g_\mu(z)/h_\mu(z)$ of the mapping Φ_n (4) of some characteristic (see Remarks 3 and 4). This means that we have a linear family of curves of degree μ, genus p and freedom $q = 1$ (see Remark 6), that is a pencil of curves $f_\mu(z) = a g_\mu(z) + b h_\mu(z)$ which are invariant under mappings Φ_n (4) and Φ_n^{-1} (5) and, consequently, are solutions of Eqs. (22) and (23). This means also that there is a set of multiplicities $\Gamma \stackrel{\text{def}}{=} \{\gamma_\alpha, \gamma'_\beta, \rho_\nu\}$ of the curve $f_\mu(z) = 0$ at F-points $O_\alpha \in \mathbf{O}^{(\text{cyc})} \cup \mathbf{O}^{(\text{int})}$, $O'_\beta \in \mathbf{O}'^{(\text{cyc})} \cup \mathbf{O}'^{(\text{int})}$ and at other indeterminacy points of nvariant $I(z)$ (ρ_ν), satisfying Eqs. (41) and (42).

Eqs. (42) are consequences of the last equation of (41). In fact, summing the third equation of (22) over α we obtain the first equation of (42) (see Remark 4 and Eqs. (6) and (8)), but squaring the third equation of (41), then summing over α and using (7), (9)–(11) and the first equation of (41) we obtain the second equation of (42). Thus, the sets of those F-points O_α, O'_β, for which multiplicities γ_α, γ'_β are not zero, are the sets $\mathbf{S}^{(\text{cyc})}$, $\mathbf{S}^{(\text{int})}$ and $\mathbf{S}'^{(\text{cyc})}$, $\mathbf{S}'^{(\text{int})}$.

Since the points O_α (O'_β) are not indeterminacy ones of the mapping $\Phi_n^{-1}(z)$ (5) $(\Phi_n(z)$ (4)), it is necessary that the orbits $\mathcal{O}_{O_\alpha}$ $(\mathcal{O}'_{O'_\beta})$ of these points $O_\alpha(O'_\beta)$ are composed without the initial points of the common indeterminacy point sets $\bar{\mathbf{S}}^{(\text{cyc})}$, $\bar{\mathbf{S}}^{(\text{int})}$, $\bar{\mathbf{S}}'^{(\text{cyc})}$ in (36) (see (35)–(38)). Let the number N of points of the set $\mathbf{S}\backslash\bar{\mathbf{S}}''^{(\text{cyc})}$ be smaller than μ^2:

$$N = \#\mathbf{S}\backslash\bar{\mathbf{S}}''^{(\text{cyc})} < \mu^2.$$

Then find the number of such cycles of the mapping $\Phi_n(z)$ (4), other than cycles $\mathbf{S}^{(\text{cyc})}$, $\mathbf{S}'^{(\text{cyc})}$, for which the total number of points of the set $\bar{\mathbf{S}}''^{(\text{cyc})}$, as the union of these cycles, equals

$$\#\bar{\mathbf{S}}''^{(\text{cyc})} = \mu^2 - N.$$

Condition (40) is necessary for integrability since otherwise the Arnold complexity, coinciding with degree $d(k)$ of the kth iteration of the mapping $\Phi_n(z)$ (4), will grow as n^k (see Definition-Remark 1 and Theorem 3). This growth corresponds to a generic mapping which is obviously not integrable. Thus we obtain a pencil of the μ-ics with the total number of free parameters (freedom) q and the genus p determined by Eqs. (44) and (43).

Due to a possible existence of some symmetry in the sets of points O_α, $O'_\beta \in \mathbf{S}$, the actual number of free parameters q_{act} may be larger than the number given by Eq. (44). The substitution of the family of curves of degree μ thus obtained into functional equations (22) and (23) gives a set of $r + 1$ linearly independent solutions of Eqs. (22) and (23) and r first integrals or invariants of the mapping under consideration. □

Two remarks follow.

Remark 9. It is obvious that, if the mapping $\Phi_n(z)$ (4) has an invariant I_μ, then it has an infinite number of algebraic invariants of the form $I'_{\mu'} = R(I_\mu)$, where R is a rational function of I_μ, moreover, all invariants depend algebraically on one of them. However, the minimal invariant is unique up to a linear-fractional change.

Remark 10. It is obvious that if we have found an integrable mapping $\Phi : z' = \Phi(z)$ and its minimal invariant I_μ, $I_\mu(\Phi(z)) = I_\mu(z)$, then we have an infinite number of integrable mappings $\Phi' : z' = \Psi^{-1} \circ \Phi \circ \Psi(z)$ of birationally equivalent to the initial one and their invariants are $I_{\mu'}(z) = I_\mu \circ \Psi(z)$.

The following theorem presents, as a corollary of Theorem 4, the method of generating integrable plane birational mappings.

Theorem 5. *Let us have a mapping in group* BirCP^2 *of characteristic* $n; i_1, \dots, i_\sigma$, *satisfying Eqs.* (6)–(11), *with generic F-points* O_α, O'_β. *Then the following recipe generates integrable mappings:*

(1) *Let us set* $\sigma_1 + \sigma_2$ *conditions of the following forms:*

$$\Phi_n^{-m_{\alpha\beta}}(O_\alpha) = O'_\beta \quad for\ \ \alpha, \beta = \alpha_1, \beta_1; \dots; \alpha_{\sigma_1}, \beta_{\sigma_1}, \tag{45}$$

$$\Phi_n^{-r_{\alpha\beta}}(O_\alpha) = O_\beta \quad for\ \ \alpha, \beta = \alpha_1, \beta_1; \dots; \alpha_{\sigma_2}, \beta_{\sigma_2}, \tag{46}$$

and analogous conditions with the change $O \leftrightarrow O'$, $m_{\alpha\beta}, r_{\alpha\beta}, \sigma_2 \leftrightarrow -(m_{\alpha\beta}, r'_{\alpha\beta}), \sigma_2$.

(2) *Let us set* σ_3 *conditions of the form*

$$\mathcal{O}^k(z) = z, \quad z \notin \mathbf{O}, \mathbf{O}' \tag{47}$$

and let us have the sets of cycles $\mathbf{C}_1: (z_1, \dots, z_{k_1}), \dots, \mathbf{C}_{\sigma_3}: (z_1, \dots, z_{k_{\sigma_3}})$. *The remaining* $\sigma - \sigma_1 - \sigma_2$ *points of the set* $\mathbf{O}$ *and* $\sigma - \sigma_1 - \sigma_2$ *points of the set* $\mathbf{O}'$ *belong to* $\mathbf{O}^{(\mathrm{inf})}$ *and* $\mathbf{O}'^{(\mathrm{inf})}$. *Then we shall form the sets* $\mathbf{S}^{(\mathrm{cyc})}$, $\mathbf{S}^{(\mathrm{int})}$, $\mathbf{S}'^{(\mathrm{cyc})}$, $\mathbf{S}'^{(\mathrm{int})}$, $\bar{\mathbf{S}}$ *according to* (35)–(38) *and construct a pencil of curves of degree* μ, *satisfying Eqs.* (39)–(42). *The substitution of the general curve of the pencil of curves* $f_\mu(z) = ag_\mu(z) + bh_\mu(z)$ *into the functional equations* (22) *and* (23) *with subsequent determination of free parameters guarantees that we have generated an integrable mapping of the characteristic under consideration with r invariants* $I_\mu(z)$ *of the form* $I_\mu(z) = g_\mu(z)/h_\mu(z)$ *for functions* g_μ, h_μ *being solutions with the same signature* $sgn(\epsilon)$ *and of the form* $I_\mu(z) = [g_\mu(z)/h_\mu(z)]^2$ *for the case of different* $sgn(\epsilon)$ (*see Theorem* 2).

5. Integrable birational quadratic plane reversible mappings with zero fixed point and their invariants, generated by invariant lines and conics

Let us give a definition of reversible mapping.

Definition 4. Let X be an arbitrary set. A one-to-one mapping $T : X \to X$ is said to be reversible if there exists another mapping $G : X \to X$ for which $T^{-1} = G \circ T \circ G$ and G is an involution: $G^2 = id$ [3,51,53].

These conditions imply that $T \circ G$ is also an involution and $T = (T \circ G) \circ G$ is the composition of two involutions. Conversely, the composition of any two involutions is reversible with respect to each of them.

To demonstrate applications of Theorems 4 and 5 for our approach, we will generate all nine integrable birational quadratic plane reversible mappings with a zero fixed point and their invariants generated by invariant lines (see Appendix A: IV and V) and invariant conics (see Appendix A: I–III, VI–IX). The characteristics of these mappings such as the

Table 1
Basic characteristics of nine integrable birational quadratic mappings (see Appendix A) with zero fixed points

N	$\Phi_2^{-k}O_1$	k	$\Phi_2^{-k}O_3$	k	$\Phi_2^{-k}O_2$	k	δ_ε	N_0	N_1	$d(m)$	μ_I	N_{inv}
I	O_1'	0	O_3'	0	O_1'	1	-1	2	1	2	2	1
II	O_3'	0	O_1'	0	O_2'	0	$+1, +1, -1$	3	0	1 or 2	1, 2, 4	3
III	O_3'	0	O_1'	0	$\in O^{(\text{inf})}$		-1	2	0	2	4	1
IV [a]	O_1	1	O_3'	0	O_2	1	$+1$	1	0	$m+1$	1, 2	2
V [a]	O_2	1	O_3'	0	O_1	1	$-1, +1$	1	0	$m+1$	2, 2	2
VI	O_1'	0	O_3'	1	$\in O^{(\text{inf})}$		-1	1	1	$m+1$	4	1
VII	O_1'	0	O_3'	1	$\in O^{(\text{inf})}$		-1	1	1	$m+1$	4	1
VIII	O_3'	1	O_1'	1	$\in O^{(\text{inf})}$		-1	0	2	$2m$	4	1
IX	O_1'	1	O_3'	1	O_2	1	$+1$	0	2	$2m$	2	1

$N_k = \#(\Phi_2^{-k}O) \cap O'$, $\Phi_2^0 \equiv id$, $k = 0, 1$, μ_I is the degree of the invariant, the value δ_ε equals: $\delta_\varepsilon = \text{sgn}(\varepsilon_g)/\text{sgn}(\varepsilon_h)$.
[a] A mapping generated by invariant straight lines, N_{inv} is the number of minimal invariants, and N is the number of the mapping in Appendix A.

Table 2
Relations between parameters in the matrix B (58)

N	1	2	3
I	$p_1 = 0$	$p_2 = 0$	$q_2 = q_3$
II	$p_3 = -p_1$	$p_2 = 0$	$q_3 = q_1$
III		$p_2 = 0$	$q_2 = q_2^*$
IV		$p_2 = -p_1$	$q_2 = q_1$
V	$p_1 = 0$	$p_2 = 0$	
VI	$p_3 = 3p_1$	$p_2 = -p_1$	$q_3 = q_2$
VII	$p_2 = p_1$	$p_3 = -3p_1$	$q_3 = q_2$
VIII	$p_2 = p_2^*$	$q_2 = q_2^{**}$	
IX	$p_3 = p_1$	$p_2 = -p_1$	$q_2 = q_2^{***}$

$q_2^* = (q_1p_3 + q_3p_1)/(p_3 + p_1)$, $p_2^* = -(p_1 + p_3)/2$, $q_2^{**} = [-q_1p_1(p_1 + 3p_3) + q_3p_3(3p_1 + p_3)]/(p_3^2 - p_1^2)$, $q_2^{***} = (q_1 + q_3)/2$.

degree $d(k)$ of dynamics of mapping Φ_2^k related to the Arnold complexity and the introduced numbers N_k of intersections of orbits $\mathcal{O}_{\mathbf{O}^{(\text{int})}}$ with the set $\mathbf{O}'^{(\text{int})}$ being the sublevels of the complexity are listed in Table 1 and the relations between the parameters of the mapping appearing from the necessary conditions for integrability of the mapping under consideration are presented in Table 2. First of all make some general comments on quadratic mappings.

Consider a general quadratic mapping $j(z) \mapsto j'(z')$, $z, z' \in \mathrm{CP}^2$ in the triangular basis (see Remark 8) with pairwise distinct F-points O_α, O_β' (the case of two or three adjacent F-points is not essential for our approach and we will consider this case elsewhere):

$$\begin{aligned}\Phi_2 : j(z) \mapsto j'(z') &= j_1'(z') : j_2'(z') : j_3'(z') \\ &= j_2(z)j_3(z) : j_1(z)j_3(z) : j_1(z)j_2(z),\end{aligned} \tag{48}$$

where j' and j are defined by linear mappings B and B_1:

$$B : z \mapsto j' = Bz, \qquad B_1 : z \mapsto j = B_1 z. \tag{49}$$

Consider a general quadratic reversible mapping with involutive symmetry between the sets **O**, **O**$'$, namely

$$B_1 = BS, \qquad S = CAC^{-1}, \qquad S^2 = id, \quad A = diag(\pm 1), \tag{50}$$

where

$$B = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix} \tag{51}$$

and C is the fundamental matrix [76] for the involutive matrix S.

Substitute $z \mapsto u = C^{-1}z$ into (50) and (49):

$$j' = B_S u, \quad j = B_S A u, \qquad B_S = BC, \tag{52}$$

where

$$\begin{aligned} A : A_1 &= diag(-1, 1, 1), \qquad A_2 = diag(1, -1, 1), \\ A_3 &= diag(-1, -1, 1), \qquad A_4 = diag(1, 1, 1), \end{aligned} \tag{53}$$

where A_4 defines the involution (see (48)–(53)). The case of a mapping with A_4 is not interesting and we shall not consider it. Mappings with A_2 and A_3 are reduced by a substitution to a mapping with $A = A_1$.

Indeed, make substitutions $u \to \tilde{v} = P_2 u$, $P_2^2 = id$, and $u \to \tilde{\tilde{v}} = \imath P_3 u$, $\quad P_3^2 = id$, where $\imath$ is the imaginary unit.

Then

$$j' = B_2 \tilde{v}, \qquad j = B_2 A_2 \tilde{v}, \qquad B_2 = BP_2, \qquad A_2 = P_2 A_1 P_2, \tag{54}$$

$$j' = B_3 \tilde{\tilde{v}}, \qquad j = B_3 A_3 \tilde{\tilde{v}}, \qquad B_3 = BP_3, \qquad A_3 = -P_3 A_1 P_3, \tag{55}$$

where the matrices P_2, P_3 are defined by

$$P_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad P_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{56}$$

Now we can make the following remark.

Remark 11. We will consider below the general quadratic mapping defined by (48), (49) and (50) with the matrices $C \equiv id$ and $A \equiv A_1$ remembering that we can always perform the changes mentioned above in integrable mappings obtained and to extend future results onto these cases.

It is clear that by no transformation $z \to u = Dz$ one can satisfy the equation $\bar{B}D = B$

$$j' = \bar{B}Dz, \qquad j = \bar{B}D(D^{-1}AD)z = \bar{B}DAz, \qquad D^{-1}AD = A, \tag{57}$$

where

$$\bar{B} = \begin{pmatrix} p_1 & q_1 & 1 \\ p_2 & q_2 & 1 \\ p_3 & q_3 & 1 \end{pmatrix}. \tag{58}$$

The general case $r_i \not\equiv 1 \ \forall i$ is to be considered separately. For simplicity we will consider below mappings with a zero fixed point, which implies $B \equiv \bar{B}$.

So, we have three principal lines J_i, J_i' and three F-points O_i, O_i', $O_i = (j_j = 0) \cap (j_k = 0)$, $(x \equiv z_1,\ y \equiv z_2,\ z \equiv z_3,\ \Lambda \equiv \Lambda_1)$,

$$j_i = -p_i x + q_i y + z, \quad j_i' = p_i x + q_i y + z, \qquad i \in (1,2,3), \tag{59}$$

$$O_i = \{q_j - q_k, p_j - p_k, p_k q_j - p_j q_k\},$$
$$i \neq j \neq k,\ i,j,k \in (1,2,3),\ \ O_i' = \Lambda O_i. \tag{60}$$

Find all integrable mappings and minimal invariants generated by invariant lines and conics. Then μ^2 equals 1 or 4 and we have to construct the set **S** of singular points of invariant I_μ. According to Theorems 4 and 5, Definition-Theorem 1 and Remark 4 for $n = 2$ we obtain that all the points O_α, O_β' are simple and

$$i_\alpha = i_\beta' = 1, \quad i_{\alpha\beta} = 1 \text{ for } \alpha \neq \beta, \quad i_{\alpha\alpha} = 0,$$
$$\sum_{\alpha=1}^{3} \gamma_\alpha = 2, \qquad \sum_{\beta=1}^{3} \gamma_\beta' = 2, \qquad \gamma_\alpha = 2 - \sum_{\beta=1,\beta\neq\alpha}^{3} \gamma_\beta'.$$

(For invariant lines we should replace 2 with 1 in these equations.) Since an irreducible conic cannot have a 2-fold point O_α, O_β', we have only two numbers γ_α, γ_β' for conics and only one number for lines (say, γ_1 and γ_3, γ_1' and γ_3' for conics and say, γ_3 and γ_3' for lines) which are other than zero and equal to 1.

Set a decomposition of the sets O, O' and consider the following conditions according to Theorems 4 and 5:

$$\Phi_2^{-k} O_i = O_j', \qquad \Phi_2^{-k} O_j = O_i', \quad i,j \in (1,3),\ \ k = 0,1, \tag{61}$$

$$\Phi_2^{-k} O_i = O_i', \qquad \Phi_2^{-k} O_j = O_j'. \quad (i \neq j) \in (1,3),\ \ k = 0,1. \tag{62}$$

Then solving these equations for general values O_i, O_j' determined by (60) and using the equations

$$\Phi_2^{-1}: \quad j_1 : j_2 : j_3 = j_2' j_3' : j_1' j_3' : j_1' j_2', \tag{63}$$

$$j_l'(O_i) = p_l(q_j - q_k) + q_l(p_j - p_k) + p_k q_j - p_j q_k, \quad i \neq j \neq k,\ l \in (1,2,3), \tag{64}$$

we obtain the relations between the parameters in the matrix $\bar{B}$ (58) for all integrable quadratic mappings with invariants generated by invariant lines and conics (see Table 2).

Substituting the following general forms for invariant curves $f_1(j(z))$, $f_2(j(z))$:

$$f_1(j(z)) = a j_1 + b j_2, \qquad f_2(j(z)) = a j_1 j_2 + b j_1 j_3 + c j_2 j_3 + d j_2^2, \tag{65}$$

into the equation for invariant curve (22) and requiring the existence of at least two linear independent solutions of this equation we have found all integrable mappings and invariants I_μ (see Appendix A: I–IX). Using Eqs. (28)–(34) from Theorem 3 we obtain the growth $d(k)$ for the degrees of integrable mappings $\Phi_2^k(z)$ (see Table 1 and Appendix A, items I–IX). Note that the mapping I follows from mapping V at $q_3 = q_2$.

6. Conclusion

As we can see from Table 1, the dynamics of an integrable mapping is determined by the numbers $k = k_{\min}$ and $N_{k_{\min}}$ which are the index and number of intersections of the orbits. In the sequel to this paper we will obtain by this method all integrable quadratic reversible mappings with invariants generated by invariant cubics, and will study their dynamics. In another paper we will consider the local and global integrability and non-integrability of known cubic polynomial Moser's mapping [64] in the framework of our approach. The theorems of this paper give us a possibility to investigate new fields such as meromorphic functions of the group BirCP^2, the integrability of the Poincare resonant systems determined by the birational mappings and others questions. It would be very interesting to analyse in the framework of our approach various relations between the conditions of the local (see, for example, the Bryuno conditions A_2, A_1', A_1'' in 9–11 [67, Theorems]) and global integrability and non-integrability, between the (algebraic) integrability and the non-algebraic one for birational (reversible) plane mappings.

As reversible mappings are qualitatively similar to symplectic mappings, it will be very useful for this analysis to exploit the enormous experience gained in the integrable and non-integrable Hamiltonian systems (see monographs [68–72]). It would also be very interesting to modify our approach by using the powerful technique of the differential forms (see monographs [73–75]) that I intend to make in one of future papers.

Let us make some comments on the famous results in the birational classification of linear systems of algebraic curves of genus p due to M. Noether, E. Bertini, G. Castelnuovo and S. Kantor (see [26, Chap. 4; 37, Theorem 7.3])

Theorem 6 [37, Theorem 7.3].

(1) *A curve of genus $p = 0$ is birationally equivalent to a line.*

(2) *An elliptic curve (of genus $p = 1$) is birationally equivalent to a cubic without multiple points.*

(3) *A hyperelliptic curve of genus p is birationally equivalent to a curve of degree $p + 2$ having a single p-fold point.*

(4) *A non-hyperelliptic curve of genus $p \geq 3$ is birationally equivalent to a normal non-singular curve (without multiple points) of degree $2p - 2$ in space $\mathrm{CP}^{(p-1)}$ unambiguously defined up to a projective transformation.*

Then in case (1) because of the pencil of rational curves is reducible with the help of some Cremona mapping to the pencil of lines the mapping Φ_n (4) having an invariant pencil of rational curves is conjugate (in the Cremona group) to the Jonquières transformation (see [24]) which maps a pencil of lines into a pencil of lines.

In case (2) it is enough due to the Bertini theorem [26] to receive an invariant pencils of curves of degree $3r$ with nine r-fold basis points (the Halphen pencil), then the remaining invariant pencils of high or other degrees and genus 1 are birationally equivalent to the Halphen pencil. Note that in the frame of modern algebraic geometry the Halphen results were repeated and supplemented in [77].

In case (3) the mapping Φ_n (4) having invariant pencil of curves of genus p is birationally equivalent to the Jonquières involution [24–26] or a composition of such an involution with a projective transformation.

In case (4) the mappings Φ_n (4) having an invariant pencil of curves of genus p are birationally equivalent (see [37]) to the involutions of finite order (periodic transformations of finite order) (see papers of Kantor [28–31], paper of Wiman [34] on finite subgroups in the Cremona group and [26, Chap. 4] about these results).

In the end we should like to point on possible applications of our results and the Kantor and Wiman results for the involutions of finite order to an investigation the k- reversible (birational) mappings (see [61–63]).

Acknowledgements

The author is deeply grateful to V.I. Arnold, V.V. Kozlov and V.B. Priezzhev for constant support, interest in the paper and useful discussions. He is also very grateful to Yu.I. Manin and V.S. Vladimirov for support. The author is deeply grateful to D.V. Anosov and S.P. Novikov for useful discussions, critical remarks and interest in the paper. He wishes to acknowledge for the interest in paper, the multiple helpful discussions with, and the manuscript reading and critical remarks by V.M. Buchstaber, V.A. Iskovskikh, I.M. Krichever, M.B. Sevryuk, A.M. Stëpin and A.N. Tyurin. The author is also thankful for useful discussions with R.I. Bogdanov, A.D. Bryuno, M.Kh. Gizatulin, V.V. Nikulin, A.D. Ponomarëv and A.P. Veselov.

Appendix A

I

$$x' = \frac{x(q_1 y + 1)}{(1 - p_3 x + q_2 y)[1 + (q_1 + q_2)y]}, \qquad y' = -\frac{y}{1 + (q_1 + q_2)y}, \tag{A.1}$$

$$I = \frac{y^2}{[(q_1 + q_2)y + 2]^2}. \tag{A.2}$$

II

$$x' = x(q_2 y + 1)D^{-1}, \qquad y' = \left(-y - \frac{p_1^2}{q_2 - q_1}x^2 - q_1 y^2\right) D^{-1},$$

$$D = 1 + (q_2 + 2q_1)y + \frac{q_1 p_1^2}{q_2 - q_1}x^2 + (q_2 + q_1)q_1 y^2, \tag{A.3}$$

$$I_1 = \frac{q_1 y + 1}{x}, \qquad I_2 = \frac{(q_1 y + 1)^2 - p_1^2 x^2 + (q_2 y + 1)^2}{x(q_2 y + 1)},$$

$$I_3 = \left[\frac{(q_1 y + 1)^2 - p_1^2 x^2 - (q_2 y + 1)^2}{x(q_2 y + 1)}\right]^2, \qquad I_3 = I_2^2 - 4(I_1^2 - p_1^2). \tag{A.4}$$

III

$$p_2 = 0, \qquad q_2 = \frac{q_1 p_3 + q_3 p_1}{p_3 + p_1}.$$

$$x' = \left(x - \frac{p_3 + p_1}{2}x^2 + q_2 x y + \frac{(q_1 - q_3)^2}{2(p_3 + p_1)}y^2\right) D^{-1}, \tag{A.5}$$

$$y' = \left(-y + \frac{p_1^2 - p_3^2}{2(q_1 - q_3)}x^2 - \frac{q_1 + q_3}{2}y^2\right) D^{-1},$$

$$D = 1 - (p_3 + p_1)x + (q_2 + q_1 + q_3)y + \frac{(p_3 q_1 - p_1 q_3)(p_3 + p_1)}{2(q_1 - q_3)}x^2$$
$$-(p_3 q_1 + p_1 q_3)xy + \frac{1}{2}[q_3(q_2 + q_1) + q_1(q_2 + q_3)]y^2,$$

$$I = \frac{[(j_2 - j_3)(j_1 - j_2)]^2}{[p_3(j_2 + j_3)(j_1 - j_2) + p_1(j_1 + j_2)(j_2 - j_3)]^2} \tag{A.6}$$
$$= \frac{\left[((q_1 - q_3)/(p_3 + p_1))^2 y^2 - x^2\right]^2}{\left[(p_3 - p_1)x^2 + \frac{4(q_1 - q_3)}{p_1 + p_3}y + \frac{(q_1 - q_3)(q_1 + q_3 + 2q_2)}{p_1 + p_3}y^2\right]^2}.$$

$$O_1 = \left(\frac{q_1 - q_3)}{(p_3 + p_1)q_2}, -\frac{1}{q_2}\right), \tag{A.7}$$

$$O_3 = \left(-\frac{q_1 - q_3)}{(p_3 + p_1)q_2}, -\frac{1}{q_2}\right), \tag{A.8}$$

$$O_2 = \left(\frac{q_1 - q_3}{p_3 q_1 - p_1 q_3}, -\frac{p_3 - p_1}{p_3 q_1 - p_1 q_3}\right). \tag{A.9}$$

Note that $O_3' = O_1$, $O_1' = O_3$. The invariant conic in the nominator of the expression for I decays to two straight lines, transforming one to another by the mapping, and the denominator is either an ellipse for $k = (q_1 - q_3)(q_1 + q_3 + 2q_2)/[(p_1 + p_3)(p_3 - p_1)] > 0$, or a hyperbola for $k < 0$, intersecting y-axis at $y = 0$ and $y = -4/(q_1 + q_3 + 2q_2)$. The fixed point is $x = y = 0$.

IV

$$x' = x(1 - p_3 x + q_3 y)D^{-1}, \quad y' = \left(-y + \frac{p_1^2 - p_3^2}{q_1 - q_3} x^2 - p_3 xy - q_1 y^2\right) D^{-1}, \tag{A.10}$$

$$\begin{aligned} D &= 1 - p_3 x + (2q_1 + q_3)y + \frac{q_1(p_3^2 - p_1^2)}{q_1 - q_3} x^2 + q_1(q_1 + q_3)y^2, \\ I_1 &= \frac{1 + q_1 y}{x} = \frac{j_1 + j_2}{j_2 - j_1}, \quad I_2 = \frac{j_1 j_2}{(j_1 - j_2)^2}, \qquad I_1^2 = 1 + 4I_2. \end{aligned} \tag{A.11}$$

V

$$x' = \frac{x + (q_1 + q_2 - q_3)xy + \frac{(q_3 - q_2)(q_3 - q_1)}{p_3} y^2}{(1 - p_3 x + q_3 y)\,[1 + (q_1 + q_2)y]}, \quad y' = -\frac{y}{1 + (q_1 + q_2)y}, \tag{A.12}$$

$$I_1 = \frac{(j_1 - j_2)^2}{(j_1 + j_2)^2} = \frac{y^2}{[2 + (q_1 + q_2)y]^2}, \qquad I_2 = \frac{(1 + q_1 y)(1 + q_2 y)}{[2 + (q_1 + q_2)y]^2}, \tag{A.13}$$

VI

$$x' = x(1 - p_1 x + q_1 y)D^{-1}, \quad y' = \left(-y - \frac{4p_1^2}{q_1 - q_2} x^2 + y p_1 x - q_2 y^2\right) D^{-1}, \tag{A.14}$$

$$\begin{aligned} D = {} & 1 - 3p_1 x + (2q_2 + q_1)y - 2p_1(q_2 + q_1)xy + q_2(q_1 + q_2)y^2 \\ & + \frac{2p_1^2(q_1 + q_2)}{q_1 - q_2} x^2, \end{aligned} \tag{A.15}$$

$$I = \left[\frac{(j_1 + j_2)(j_2 - j_3)]^2}{3j_2(j_1 - j_3) + j_1 j_3 - j_2^2}\right]^2 = \frac{[2p_1^2 x^2 + (q_1 - q_2)y(q_2 y + 1)]^2}{[(q_1 + q_2)y + 2]^2 x^2}. \tag{A.16}$$

VII

$$x' = x(1 - p_1 x + q_1 y)D^{-1}, \qquad y' = -y(1 + 3p_1 x + q_2 y)D^{-1}, \tag{A.17}$$

$$\begin{aligned} D &= 1 + p_1 x + (q_1 + 2q_2)y + 2p_1(q_1 + q_2)xy + q_2(q_1 + q_2)y^2, \\ I &= \frac{[3(j_1 + j_3)j_2 + (j_1 j_3 + j_2^2)]^2}{[(j_1 - j_2)(j_2 - j_3)]^2} \\ &= \mathrm{const} \frac{[-2p_1^2 x^2 + (q_1 + 3q_2)y + q_2(q_1 + q_2)y^2]^2}{y^2 x^2}, \end{aligned} \tag{A.18}$$

VIII

$$p_2 = -\frac{p_1 + p_3}{2}, \qquad q_2 = \frac{-q_1 p_1(p_1 + 3p_3) + q_3 p_3(3p_1 + p_3)}{p_3^2 - p_1^2}$$

$$x' = \left(x + \frac{p_3 q_1 + p_1 q_3}{p_3 + p_1} xy - \frac{2p_1 p_3}{p_3 + p_1} \frac{(q_3 - q_1)^2}{(p_3 - p_1)^2} y^2\right) D^{-1},$$

$$y' = \left(-y + \frac{(p_3 - p_1)(p_3 + p_1)}{2(q_3 - q_1)}x^2 + \frac{p_3 + p_1}{2}yx - \frac{q_3 p_3 - p_1 q_1}{p_3 - p_1}y^2\right) D^{-1},$$

$$D = 1 - \frac{p_3 + p_1}{2}x + (q_1 + q_3 + q_2)y - \frac{(q_3 p_3 - p_1 q_1)(p_3 + p_1)}{2(q_3 - q_1)}x^2$$
$$-\frac{1}{2}(q_2(p_3 + p_1) + q_3 p_1 + p_3 q_1)yx$$
$$+\frac{q_3 p_3 - p_1 q_1}{p_3 - p_1}\left(q_2 + q_3 + q_1 - \frac{q_3 p_3 - p_1 q_1}{p_3 - p_1}\right)y^2, \tag{A.19}$$

$$I = \frac{[(p_3^2 + 6p_1 p_3 + p_1^2)(j_2'^2 + j_1' j_3') + 2(p_1 + p_3)^2 (j_1' + j_3') j_2']^2}{[(p_3 - p_1)(j_1' j_3' - j_2'^2) + 2(p_1 + p_3) j_2' (j_1' - j_3')]^2}$$
$$= \left[-\frac{(p_3 - p_1)^2}{4}x^2 + \left(q_2^2 + q_1 q_3 - \frac{2(p_3 + p_1)^2 (q_3 - q_1)^2 p_3 p_1}{(p_3^2 - p_1^2)^2}\right)y^2\right.$$
$$\left. + (2q_2 + q_1 + q_3)y + 2\right]^2$$
$$\times \frac{1}{\left[\frac{p_3 - p_1}{4}x^2 - \frac{(q_3 - q_1)[q_3 p_3 (p_3 + 2p_1) - q_1 p_1 (p_1 + 2p_3)]}{(p_3^2 - p_1^2)(p_3 + p_1)}y^2 - \frac{q_3 - q_1}{p_3 + p_1}y\right]^2}. \tag{A.20}$$

IX

$$x' = \left(x - p_1 x^2 + \frac{q_1 + q_3}{2}xy + \frac{(q_1 - q_3)^2}{8p_1}y^2\right) D^{-1},$$
$$y' = -y\left(1 + p_1 x + \frac{q_1 + q_3}{2}y\right) D^{-1}, \tag{A.21}$$

$$D = 1 - p_1 x + \frac{3}{2}(q_1 + q_3)y + \frac{(3q_1 + q_3)(q_1 + 3q_3)}{8}y^2,$$
$$I = \frac{(j_2 + j_3)(j_1 + j_2)}{(j_1 - j_2)(j_2 - j_3)} = \frac{\left(\frac{1}{4}(q_1 + 3q_3)y + 1\right)\left(\frac{1}{4}(3q_1 + q_3)y + 1\right)}{\frac{1}{16}(q_1 - q_3)^2 y^2 - p_1^2 x^2} \tag{A.22}$$

$$O_1 = \left(\frac{q_1 - q_3}{(q_1 + 3q_3)p_1}, -\frac{4}{q_1 + 3q_3}\right), \qquad O_2 = \left(\frac{1}{p_1}, 0\right),$$
$$O_3 = \left(-\frac{q_1 - q_3}{(3q_1 + q_3)p_1}, -\frac{4}{3q_1 + q_3}\right). \tag{A.23}$$

This mapping has three invariant straight lines and one conic reducible into straight lines:

(1) $y = 0$, $(O_2' O_2)$,

(2) $y = -4/(q_1 + 3q_3)$, $(O_1 O_1')$,

(3) $y = -4/(3q_1 + q_3)$, $(O_3 O_3')$.

The set of fixed points of this mapping is the conic $\Delta = \Delta_1\Delta_2$ (see Definition 2), reducible into two straight lines Δ_1, Δ_2, where Δ_1 is the line of fixed points (y = const, $x \to \infty$, in the variables $\tilde{y} = 1/y, \tilde{x} = 1/x : \tilde{x} = 0,\ \tilde{y}$ = const); Δ_2: $y = 0$ is the line of fixed points.

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